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ABSTRACT

We explore here the possible importance of time-dependent convection electric fields as an agent for diffusing trapped magnetospheric radiation inward toward the earth. Using a formalism (Birmingham, Northrop, and Fälthammar, 1967) based on first principles, and adopting a simple model for the magnetosphere and its electric field, we succeed in deriving a one dimensional diffusion equation to describe statistically the loss free motion of mirroring particles with arbitrary but conserved values of the first two adiabatic invariants M and J. Solution of this equation bears out the fact that reasonable electric field strengths, correlated in time for no longer than the azimuthal drift period of an average particle, move particles toward the earth at a rate at least an order of magnitude faster than electric fields whose source is a fluctuating current on the magnetopause.

CONVECTION ELECTRIC FIELDS AND THE DIFFUSION OF TRAPPED MAGNETOSPHERIC RADIATION

INTRODUCTION

There has been growing rapidly a body of literature concerned with the theory of interaction of magnetospheric particles with collective oscillations of the magnetospheric medium (Wentzel, 1961; Parker, 1961; Dragt, 1961; Dungey, 1963; Cornwall, 1964; Chang and Pearlstein, 1965; Kennel and Petschek, 1966; Cocke and Cornwall, 1967; Roberts and Schulz, 1968). The thesis of this work is that at heights where interparticle collisions are infrequent, wave-particle interactions are an effective agent for depleting the energetic trapped population: successive wave induced scatters of a particle's pitch angle cause it to mirror at altitudes increasingly closer to the earth, until ultimately the particle reaches the collision dominated level of the ionosphere where its mirror motion is disrupted. Impressive corroboration of experiment with theory is presented by Kennel and Petschek (1966) who show that for a range of L-values (>4) equatorial fluxes of both energetic protons and electrons lie near values expected theoretically when particles and waves are in quasi-linear equilibrium.

The existence of steady-state particle fluxes in the presence of such losses obviously requires a balancing input. We here consider in detail a version of the theory that particles are injected at the magnetopause and then diffuse in L, the equatorial crossing distance of a magnetic field line expressed in numbers of earth radii R_E.

Diffusion in L is likewise a topic which has received extensive attention. An exhaustively explored (Kellogg, 1959; Parker, 1960; Davis and Chang, 1962; Tverskoy, 1964; Dungey, 1965; Nakada and Mead, 1965; Conrath, 1967) idea has been that L-diffusion is driven by a time varying electric field which arises from fluctuations in the currents separating the terrestrial and interplanetary magnetic fields. The calculated particle diffusion rate is generally sufficient to balance interparticle scattering losses (Nakada and Mead, 1965; Newkirk and Walt, 1968) but appears grossly inadequate when wave particle interaction losses are taken into account (Kavanagh, 1968).

Cladis (1966) has proposed a slightly different mechanism: at very low L-values ionospheric current variations produce fluctuating electric fields which drive the diffusion process. The effectiveness of such a mechanism diminishes, however, as a function of altitude in the magnetosphere.

Somewhat in contrast, one of Cornwall's (1968) speculations is that L-diffusion is a magnetospheric manifestation of Bohm diffusion (1949), an anomalously large diffusion rate across magnetic field lines observed in many laboratory plasma experiments (e.g., cf. Pease et al., 1966). It is thought that this enhancement above the level of classical, collisional diffusion is the result of particles interacting with turbulence in the plasma. The turbulence in turn is a consequence of the fact that laboratory plasmas have density and/or temperature gradients and hence are susceptible to the growth of unstable drift waves (cf. Kadomtsev, 1965). In the magnetosphere similar gradients exist, and it is possible that Bohm diffusion is operative, particularly in the vicinity of the plasmapause.

Our own viewpoint is in line with the thinking of Fälthammar, who has continually stressed the importance of electrostatic fields ($\nabla \times \mathbf{E} = 0$), varying in time and of magnetospheric spatial dimensions, in the dynamics of trapped radiation. Specifically, we consider as the driving force in L-diffusion the electrostatic field

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V = -\frac{\mathbf{v}_f \times \mathbf{B}}{c} \tag{1}$$

associated with the ideal hydromagnetic flow $\mathbf{v}_f$ of low energy magnetospheric plasma. (Our demarcation between low and high energy particles is roughly 5 keV, an energy at which gradient and curvature drifts become important components of the guiding center motion at magnetospheric heights for typical electric fields. Low energy particles essentially follow the hydromagnetic convection pattern and are of interest to us only as the source of the polarization electric field.) Fälthammar (1965), Brice (private communication), Obayashi and Nishida (1968), Kavanagh (1968), and Cornwall (1968) have each suggested the possible importance of the convection electrostatic field to the diffusion of energetic particles.

In this paper we adapt the diffusion theory developed from first principles by Birmingham, Northrop, and Fälthammar (1967) to a simple model of the magnetosphere and its impressed convection electric field. No restriction to equatorial particles is made. The one dimensional diffusion equation which statistically describes loss-free particle motion in this model is then solved. Results

are compared with those from a model in which magnetopause currents are the source of the magnetospheric electric field. We conclude that convection electric fields move particles radially at a rate an order of magnitude faster than the electric fields arising from magnetopause currents.

Unless otherwise specified, Gaussian (cgs.) units are used throughout this paper.

MODEL

We consider the following elementary model of the earth's magnetosphere. The magnetic field **B**, constant in time, is due solely to a magnetic dipole of moment μ at the origin of an r, ϑ, ϕ spherical coördinate system. In the conventional sense, μ is aligned with the south normal to the ecliptic plane, colatitude ϑ is measured from the $-\mu$ axis, and azimuth ϕ is measured clockwise from the antisolar meridian (i.e., $\phi = \pi/2$ on the dawn meridian). Modification of the magnetic field by either the energetic particles or the cold plasma component is neglected in this low-β model.

The electric field **E** is irrotational, variable in time, and (from Equation 1) everywhere perpendicular to **B**. It has the feature, essential for L-diffusion, that it is asymmetric about an energetic particle's longitudinal drift path in the dipole field. We describe **E** by the potential V

$$V = \frac{A(t)\, r}{\sin^2 \vartheta} \sin\phi , \tag{2}$$

A being a positive, time-dependent amplitude. The form Equation (2) is the fundamental ($m = 1$) asymmetric mode in Fälthammar's (1965) Fourier expansion of a general longitudinally dependent potential. Since $r \sin^{-2} \vartheta$ and ϕ are both constant on dipole field lines, B lines are equipotentials and $\mathbf{E} \cdot \mathbf{B}$ is zero. In the $\vartheta = \pi/2$, equatorial plane of the dipole, the electric field derived from Equation 2 has the magnitude A and is uniformly directed, dawn to dusk.

In this simple model, low energy plasma in the equatorial plane flows in straight lines (oriented along $\phi = \pi$) from the night to the day side of the magnetosphere. The model thus crudely represents (in a mathematically tractable manner) the flow pattern as depicted by Levy et al. (1964) for Dungey's (1961) field line merging model of the magnetosphere (cf. also Petschek, 1964; Brice, 1967). Except for regions near the magnetopause our model flow is also crudely representative of the pattern envisaged by Axford and Hines (1961) in their closed magnetosphere model. (We argue further that by allowing A in Equation 2 to reverse sign beyond a certain large $L = L_R$, an even better approximation to the Axford and Hines picture is obtained. The diffusion of energetic particles in our analysis depends only on the autocorrelation of A. Thus, as long as the statistical properties of A are the same for all L, our results are valid even for $L > L_R$.)

We treat A as a fluctuating quantity with an average magnitude of 4×10^{-4} V/m, a value typical of hydromagnetic models of the magnetosphere. The fluctuations reflect similar variations in the intensity of the solar wind, to which the magnetospheric flow pattern is directly coupled. A one hour time scale seems

(Carpenter and Stone, 1967; Obayashi and Nishida, 1968; Brice, private communication) appropriate for fluctuations in A(t). We shall find that the mean square fluctuation amplitude $\mathcal{Q} = \langle (\delta A)^2 \rangle$ and the autocorrelation time τ_c of the fluctuations are quantities which scale the diffusion time of the energetic particles.

Our simple model takes no account of the electric field associated with the rotation of a finite radius, conducting earth. In hydromagnetic models of the magnetosphere, the rotation electric field is considered to dominate the dynamics of low energy plasma at radial distances as far out as the plasmapause. Some evidence exists (Nishida, 1966; Carpenter and Stone, 1967), however, that the pattern of outer magnetosphere electric fields persists, though perhaps in a recessive role, in the region dominated by rotation. The electric field associated with the rotation of the earth is both nearly longitudinally symmetric and nearly constant in time. (The tilt of the earth's magnetic axis with respect to its spin axis introduces a 24 hour periodicity into the rotation electric field. Such a time variation is too slow to be consequential in the diffusion of energetic particles.) For L-diffusion the rotation field plays no significant role.

We realize fully that the model of electric and magnetic fields adopted in this paper lacks the detail of extant qualitative hydromagnetic models. However, for our purposes — an <u>analytical</u> study of L-diffusion — we feel that the present model extracts the salience of its more complicated counterparts and simultaneously affords a simplicity necessary for achieving mathematical results.

THE DIFFUSION EQUATION FOR ENERGETIC PARTICLES

For energetic particles in our model, the ratio of particle gyro-radius to the scale length ℓ (typically $1R_E$) of electric and magnetic fields is small:

$$\epsilon = \frac{a}{\ell} \approx 4L^3 \sqrt{Wm} \times 10^5 \ll 1 \ . \qquad (3)$$

Here W is the energy (in eV) of an energetic particle of mass m (in grams) at a distance LR_E from the earth. Further, the typical one hour time variation of the electric field lies on the slow time scale of the azimuthal drift of these particles in the dipole magnetic field. Finally we note that the ratio E/B of the field magnitudes, electric to magnetic, is at least $O(\epsilon)$ in smallness.

Northrop's (1963) systematic development of the adiabatic motion of charged particles asserts that for the conditions stated in the preceding paragraph, energetic particles move in such a fashion that the first two adiabatic envariants of their motion — the magnetic moment $M = mv_\perp^2/2B$ and the longitudinal invariant $J = m\oint v_\parallel \, ds$ — are conserved. (Here $\perp$ and $\parallel$ denote velocity components perpendicular and parallel to the dipole magnetic field, and the s-integral is extended over a complete guiding center bounce path.) The third or flux invariant Φ is not preserved in the present treatment.

Magnetic moment conservation implies that particles moving inward in the dipole magnetic field gain kinetic energy. A rough estimate based on both M and J conservation

$$\frac{M^{1/2}}{J} = \text{const } \alpha \frac{(v_\perp)_{eq}}{(v_\parallel)_{eq}} \frac{L^{3/2}}{L} = L^{1/2} \operatorname{ctn} \lambda_{eq} \qquad (4)$$

also indicates that inward moving particles flatten their equatorial pitch angles λ_{eq}. In doing so, such particles contribute to a pitch-angle anisotropy which can drive microinstabilities and lead to diffusion into the loss cone of the magnetic mirror.

All particles with the same values of the invariants M and J and with guiding centers lying on the same dipole field line (we here identify a dipole line by the constant values of the Euler potentials, $\beta = \phi$, $\alpha = -\mu \sin^2 \vartheta / r = -\mu / LR_E$, on it) have practically the same experience over the course of one bounce, since the electric field is changing on the much longer drift time scale. In our treatment we consider a number of (identical) bounce-averaged guiding centers with invariants M and J as closely representing the actual M, J particles gyrating about and bouncing along the line α, β. Equations for such bounce averaged guiding centers were first obtained by Northrop and Teller (1960), who performed a bounce average of the ordinary guiding center equations of motion.

Within the framework of bounce-averaged guiding center theory our treatment of the energetic particles is a statistical one: we adapt to the present model a general theory (Birmingham, Northrop, and Fälthammar, 1967, hereafter referred to as BNF) for the diffusion of guiding centers, the diffusion in the BNF theory being due to small electromagnetic fluctuations which conserve M and J. Only statistical properties of the fluctuations are assumed to be known. In the present model, of course, the magnetic field is static and the BNF treatment simplifies greatly.

In Appendix A adaptation of the BNF theory to the present model is carried out and the equation

$$\frac{\partial\langle\overline{Q}\rangle(\alpha, M, J, t)}{\partial t} = \frac{\partial}{\partial \alpha}\left[\overline{D_{\alpha\alpha}}\,\frac{\partial\langle\overline{Q}\rangle}{\partial \alpha}\right] \tag{5}$$

is derived. Here $\langle\overline{Q}\rangle$ is the β-average of the guiding center density $\langle Q\rangle(\alpha, \beta, M, J, t)$ defined for a four dimensional — α, β, M, J — phase space:

$$\langle\overline{Q}\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\beta\, \langle Q\rangle\,(\alpha, \beta, M, J, t) \tag{6}$$

$\langle Q\rangle$ itself is a statistical average, this average (denoted by $\langle\ \ \rangle$) being carried out for the same set of M, J particles over an ensemble of fluctuation systems. The β-average is carried out at constant α, M, J, and t. Use of the definition $\alpha = -\,\mu/LR_E$ and the relationship

$$\langle\overline{Q}\rangle(\alpha, M, J, t) = \frac{dL}{d\alpha}\, n(L, M, J, t) = \frac{L^2\, R_E}{\mu}\, n(L, M, J, t) \tag{7}$$

leads to the following more familiar form of Equation 5,

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial L}\left[f(L)\,\frac{\partial}{\partial L}\left(nL^2\right)\right], \tag{8}$$

where

$$f(L) = \frac{R_E^{\,2}}{\mu^2}\, L^2\, \overline{D_{\alpha\alpha}\,(L)}\,. \tag{9}$$

While Equation 8 is identical in form with the equation generally introduced in L-diffusion studies, it is here valid without the restriction to $J = 0$ particles inherent in most past work.

Were there no electric field, M, J particles would drift longitudinally at constant α in the dipole field with an angular drift frequency $\omega_D (\alpha, M, J)$. Because of the fluctuating electric field, however, such M, J particles drift across α, changing their kinetic energy in the process. Equations 5 and 8 describe this diffusion across dipole field drift shells, all gyro-phase, bounce-phase, and longitude dependence having been averaged from the problem at the present level of description.

Northrop (private communication) has shown the following relationship between $\langle Q \rangle (\alpha, \beta, M, J, t)$ and the particle flux j differential in kinetic energy W and solid angle Ω:

$$\langle Q \rangle = \frac{\pi}{2} \frac{j}{W} \cdot \tag{10}$$

Since $\langle Q \rangle$ is independent of arc length s along each field line, j/W is similarly s-independent. At each value of s along a line α, β, however, the flux j is to be measured for a different solid angle Ω and a different energy W, Ω and W being determined by the conservation of M and J on that field line.

The diffusion coefficient $\overline{D_{\alpha\alpha}}$ is derived in terms of δV, the fluctuating part of the potential, in a straightforward manner from BNF results:

$$\overline{D_{\alpha\alpha}}(\alpha, J, M, t) = \frac{c^2}{2\pi}\int_0^{2\pi} d\beta \left\langle \int_0^{\infty} d\tau \left(\frac{\partial(\delta V)}{\partial \beta}\right)_{\alpha,\beta-\omega_D\tau,\, t-\tau} \left(\frac{\partial(\delta V)}{\partial \beta}\right)_{\alpha,\beta,t} \right\rangle . \quad (11)$$

Subscripts here indicate the functional dependence of the quantities involved; thus $(\partial\delta V/\partial\beta)_{\alpha,\beta,t}$ is a function $F(\alpha, \beta, t)$ while $(\partial\delta V/\partial\beta)_{\alpha,\beta-\omega_D\tau,t-\tau}$ is $F(\alpha, \beta-\omega_D\tau, t-\tau)$.

Using Equation 2 and the definition of α and β we further obtain

$$\overline{D_{\alpha\alpha}} = \frac{c^2}{2\pi}\frac{\mu^2}{\alpha^2}\int_0^{2\pi} d\beta \left\langle \int_0^{\infty} d\tau\, \delta A(t-\tau)\, \delta A(t) \cos(\beta - \omega_D \tau) \cos\beta \right\rangle$$

$$= \frac{c^2}{2}\frac{\mu^2}{\alpha^2}\int_0^{\infty} d\tau \, \langle \delta A(t-\tau)\, \delta A(t)\rangle \cos\omega_D \tau . \quad (12)$$

Dependence of $\overline{D_{\alpha\alpha}}$ on M and J for this case where magnetic field lines are equipotentials occurs only through ω_D

$$\omega_D(\alpha, M, J) = \frac{2c}{e\alpha} MB_m(\alpha, M, J)\left[\frac{3}{2} - \frac{J}{4MB_m T(\alpha, M, J)}\right] . \quad (13)$$

Equation 13, a form due to Northrop (1966), gives ω_D for a particle with invariants M and J whose guiding center is on dipole line α at any β, in terms of M, J, α, B_m, and T. Here B_m and T are respectively the mirror magnetic field strength and the bounce period of such a particle.

In order that the longitudinal invariant J exist, E must be at least $O(\epsilon)$ in smallness compared with the $O(1)$ magnetic field; as in BNF, we consider here the yet more restricted case where E is $O(\epsilon\delta)$, $\epsilon \ll \delta \ll 1$. With this ordering and the form of the coefficient as given by Equation 12, our diffusion equation, Equation 5, is correct through $O\left(\epsilon\delta^2\right)$.

In passing we note that

$$\overline{D_{\alpha\alpha}} = \left\langle \overline{\frac{(\Delta\alpha)^2}{\Delta t}} \right\rangle = \frac{\mu^2}{L^4 R_E^2} \left\langle \overline{\frac{(\Delta L)^2}{\Delta t}} \right\rangle = \frac{\mu^2}{L^4 R_E^2} \overline{D_{LL}} , \tag{14}$$

so that, using Equation 12,

$$\overline{D_{LL}} = \frac{c^2}{2} \frac{L^6 R_E^4}{\mu^2} \int_0^\infty d\tau \, \langle \delta A(t-\tau) \, \delta A(t) \rangle \, \cos \omega_D \tau \, . \tag{15}$$

Equation 15 exhibits a familiar dependence (Fälthammar, 1965; Cornwall, 1968) on a) the sixth power of L (explicitly) and b) the value of the power spectrum of the fluctuating part of the equatorial electric field at the resonant frequency $\omega = \omega_D$.

A reasonable direction to proceed, in view of the paucity of direct experimental evidence of electric fields and their time variations, is to assume that the autocorrelation $\langle \delta A(t-\tau) \, \delta A(t) \rangle$ has the form

$$\langle \delta A(t-\tau) \, \delta A(t) \rangle = \mathcal{A} \exp - \frac{\tau^2}{\tau_c^2} \tag{16}$$

stationary in time (independent of t). This choice is similar to one made by Cornwall (1968) and is here motivated by the idea that the equatorial electric field A exhibits no periodic amplitude variations, is always directed from dawn to dusk, and is random on the time scale on which the solar wind executes time variations of large spatial extent. (The correlation time τ_c is thus typically one hour.)

Substituting Equation 16 into Equation 12 and integrating, we obtain

$$\overline{D_{\alpha\alpha}} = \frac{c^2 \mu^2}{4\alpha^2} \sqrt{\pi}\, \tau_c\, \mathcal{A} \exp - \frac{\omega_D^2 \tau_c^2}{4}. \tag{17}$$

Given a value of τ_c, note that $\overline{D_{\alpha\alpha}}$ for fast particles with short drift periods $\tau_D = 2\pi/|\omega_D| \ll \tau_c$ is <u>exponentially small</u>. Such particles experience an essentially time independent ensemble average electric field which drives them to smaller α for one half of their azimuthal drift period and to larger α for the other half; over a complete drift period the two effects cancel.

For slower particles, $\tau_D \gg \tau_c$, we can represent $\overline{D_{\alpha\alpha}}$ approximately as

$$\overline{D_{\alpha\alpha}} \approx \frac{c^2 \mu^2}{4\alpha^2} \sqrt{\pi}\, \tau_c\, \mathcal{A} \tag{18}$$

<u>independent of a particle's M and J</u>. In the following section we shall consider the solution to the diffusion Equation 5 with Equation 18 as the diffusion coefficient. Note from Equation 13 that the approximation $\tau_D = 2\pi/|\omega_D| \gg \tau_c$ restricts M, J particles which can be effectively diffused to an energy range determined

from

$$\frac{\pi}{c} \frac{|e|\,\mu}{LR_E} \frac{1}{W\left(\frac{3}{2} - \frac{J}{4WT}\right)} \gg \tau_c \ . \tag{19}$$

For given W and L, the greatest restriction is on J = 0 particles.

SOLUTION OF THE LOSS-FREE DIFFUSION EQUATION

Pursuant to the discussion in the previous section we now write down the equation

$$\frac{\partial \langle \overline{Q} \rangle}{\partial t} = \frac{\partial}{\partial \alpha} \left[\left(\frac{c^2 \mu^2}{4\alpha^2} \sqrt{\pi}\, \tau_c\, \alpha \right) \frac{\partial \langle \overline{Q} \rangle}{\partial \alpha} \right] \tag{20}$$

which describes the loss free diffusion of particles whose conserved M and J satisfy (19) at all α.

The general solution to Equation 20 regular at $\alpha = 0$ ($L = \infty$) and $\alpha = -\infty$ ($L = 0$) for the initial condition $\langle \overline{Q} \rangle (\alpha,\ t = 0) = \psi(\alpha)$ can be worked out in a straightforward manner. We refer the reader interested in the mathematical steps leading to this solution to Parker's (1960) paper and here merely quote the result

$$\langle \overline{Q} \rangle = \frac{|\alpha|^{3/2}}{2} \int_0^\infty dx\, x \exp\left(-x^2 \tau\right) J_{3/4}\left(\frac{x\alpha^2}{2}\right) \left\{ \int_0^\infty dy\, y^{3/2} J_{3/4}\left(\frac{xy^2}{2}\right) \psi(-y) \right\} \ . \tag{21}$$

In Equation 21 $J_{3/4}$ is the Bessel Function of order 3/4 and $\tau = \sqrt{\pi}\, c^2 \mu^2 \tau_c\, \mathcal{Q} t/4$ is normalized time. Note that τ depends linearly on both the correlation time τ_c and $\mathcal{Q}$, seen from Equation 16 to be the (time independent) mean square fluctuation $\left\langle [\delta A(t)]^2 \right\rangle$ of A.

The choice of $\psi(\alpha) = N\delta(\alpha - \alpha_i)$, corresponding to a $\langle \bar{Q} \rangle$ initially spiked at $L_i = -\mu/\alpha_i R_E$, facilitates integration of Equation 21. The time development of $\langle \bar{Q} \rangle$ is then essentially the same as if ψ were a Gaussian whose dispersion σ^2 is much smaller than L_i^2.

Using the δ-function form of ψ, we perform the integrations in Equation 21 and obtain

$$\langle \bar{Q} \rangle = N \frac{|\alpha|^{3/2} |\alpha_i|^{3/2}}{4\tau} \exp - \left(\frac{\alpha^4 + \alpha_i^4}{16\tau} \right) I_{3/4} \left(\frac{\alpha^2 \alpha_i^2}{8\tau} \right) . \tag{22}$$

$I_{3/4}$, the 3/4 order Bessel function of imaginary argument, has the simple integral representation (Gradshteyn and Ryzhik, 1965)

$$I_{3/4}(z) = \left(\frac{z}{2}\right)^{3/4} \frac{1}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{+1} (1 - x^2)^{1/4} \exp(zx)\, dx , \tag{23}$$

Γ being the usual gamma function. With $I_{3/4}$ expressed in integral form and $|\alpha|$ identified as μ/LR_E, $\langle \bar{Q} \rangle$ has the form

$$\langle \bar{Q} \rangle = \frac{N}{32\,\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)} \frac{\mu^6}{R_E^6} \frac{1}{L_i^3 L^3 \tau^{7/4}} \exp - \left[\frac{\mu^4}{16 R_E^4 \tau} \left(\frac{1}{L_i^4} + \frac{1}{L^4} \right) \right]$$

$$\int_{-1}^{+1} dx\, (1 - x^2)^{1/4} \exp\left(\frac{\mu^4}{8 R_E^4 L_i^2} \frac{x}{L^2 \tau} \right) . \tag{24}$$

The L, τ dependence of $\langle\overline{Q}\rangle$ has been investigated numerically for a case in which the initial position of the δ-function is $L_i = 8$. The results appear in Figure 1: $\langle\overline{Q}\rangle$ is in arbitrary units and $T = 2.4 L_i^4 R_E^4 \tau/\mu^4 = 1.4 \times 10^6 (\tau_c \mathcal{A})\, t$ [t and τ_c are in hrs., $\mathcal{A}$ in $(V/m)^2$. Values $\mu/R_E^3 = .3$ gauss, $L_i = 8$, and $c = 3 \times 10^{10}$ cm/sec have been used in relating t and T.] For $\tau_c = 1$ hr., $\mathcal{A} = (2 \times 10^{-4}\ V/m)^2$, values which we feel appropriate for the present model, T = 1 unit thus corresponds to approximately 18 hrs. Also, for these values the diffusion coefficient $\overline{D}_{LL} = 1.5 \times 10^{-4} L^6 R_E^2/\text{day}$.

Note that the peak value of $\langle\overline{Q}\rangle$ moves toward decreasing radius as time progresses. Asymptotic expressions for L_{max}, the position of maximum $\langle\overline{Q}\rangle$, are derived in Appendix B. For $L_i = 8$ and in terms of the parameter T, these expressions are

$$L_{max} = 8\left[1 - .42\,T + O\left(T^2\right)\right] \qquad T \ll 1\ , \tag{25a}$$

$$L_{max} = \frac{8}{(5\,T)^{1/4}}\left[1 - \frac{.02}{T} + O\left(T^{-2}\right)\right] \qquad T \gg 1\ . \tag{25b}$$

For each value of T in Figure 1, the position of L_{max} is accurately predicted by Equation 25b.

Note further that the drop-off of $\langle\overline{Q}\rangle$ with L is more rapid on the low L side of the maximum than on the high L side. This feature is a consequence of the fact that the particle $\mathbf{E} \times \mathbf{B}$ drift velocity becomes progressively slower (owing to its 1/B dependence) as the dipole magnetic field source at L = 0 is approached:

particles on the leading inner edge of $\langle\bar{Q}\rangle$ have the most difficulty executing radial drifts.

On the other hand, our model is quite unrealistic at large L, where the combination of a rapidly decreasing magnetic field and a radially constant (though ϑ, ϕ dependent) electric field yields unbounded electric drifts. As a result of such drifts there is a finite flux of particles to $L = \infty$ and the total number of M, J particles at all α is not conserved; i.e., from Equations 20 and 24

$$\frac{\partial}{\partial\tau}\int_{-\infty}^{0}\langle\bar{Q}\rangle\, d\alpha = \frac{L^4 R_E^3}{\mu^3}\frac{\partial\langle\bar{Q}\rangle}{\partial L}\Bigg|_0^\infty$$

$$= -\frac{3N}{32\,\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}\frac{\mu^3}{R_E^3}\frac{1}{L_i^3}\frac{1}{\tau^{7/4}}\exp -\frac{\mu^4}{16\,R_E^4\,L_i^4}\int_{-1}^{+1} dx\,\left(1-x^2\right)^{1/4} < 0. \quad (26)$$

Aware of, but untroubled by, this fact we feel that proper modification of the model at large L would result in a particle conserving $\langle\bar{Q}\rangle$ essentially the same as Equation 24 at L-values of interest.

Further feeling for the diffusion rate is gained by calculating the velocity with which L_{max} moves inward. From Equations 25 we obtain

$$\frac{dL_{max}}{dt} = -3.4\,\frac{dT}{dt}\left[1+O(T)\right] \qquad (T \ll 1)$$

$$= -4.8\times 10^6\,\tau_c\,\mathfrak{a}\left[1+O(t)\right]\frac{R_E}{hr} \qquad \left(t \ll \frac{0.7\times 10^{-6}}{\tau_c\,\mathfrak{a}}\ hrs\right), \quad (27a)$$

$$\frac{dL_{max}}{dt} = -\frac{2}{(5)^{1/4}(T)^{5/4}}\frac{dT}{dt}\left[1+O(T^{-1})\right] \qquad (T \gg 1)$$

$$= -\frac{.039}{\left(\tau_c \mathcal{Q}\right)^{1/4} t^{5/4}}\left[1+O\left(t^{-1}\right)\right]\frac{R_E}{hr} \qquad \left(t \gg \frac{0.7\times 10^{-6}}{\tau_c \mathcal{Q}}\ hrs\right). \tag{27b}$$

In Equations 27 t and τ_c are again to be expressed in hours and $\mathcal{Q}$ in $(V/m)^2$.

By way of comparison we have also solved the loss free diffusion equation for $\langle \bar{Q} \rangle$ using the coefficient

$$\overline{D_{LL}} = 0.031\, L_b^2 \left(\frac{L}{L_b}\right)^{10} \frac{(\text{earth radii})^2}{\text{day}} = 3.6 \times 10^{-7}\, L_b^2 \left(\frac{L}{L_b}\right)^{10} \frac{(\text{earth radii})^2}{\text{sec}} \tag{28}$$

calculated by Nakada and Mead (1965) for $J = 0$ particles diffusing under the effect of electric fields associated with sudden magnetic commencements. In this equation L_b is the L value of the quiet time boundary at the sub-solar point in the Mead (1964) model of the magnetosphere (We take $L_b = 10.$). The Nakada-Mead diffusion coefficient is used as a value representative of theories in which the driving electric fields arise from current variations on the magnetopause. For $L_b = 10$, the ratio of this coefficient to that $\overline{D}_{LL}$ associated with a typical convection electric field is $2 \times 10^{-6} L^4$.

Relating $\overline{D_{LL}}$ as given by Equation 28 to $\overline{D_{\alpha\alpha}}$ via Equation 14, and using the definition $L = -\mu/\alpha R_E$, we find for this case

$$\overline{D_{\alpha\alpha}} = 3.6 \times 10^{-15} \frac{\mu^8}{\alpha^6 R_E^8}. \tag{29}$$

The diffusion equation in terms of normalized time τ' is thus

$$\frac{\partial\langle\bar{Q}\rangle}{\partial\tau'} = \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^6}\frac{\partial\langle\bar{Q}\rangle}{\partial\alpha}\right). \tag{30}$$

For the same δ-function initial condition and in the same notation used previously, the solution to Equation 30 is

$$\langle \overline{Q} \rangle = \frac{N}{(256)\,(2)^{1/4}\,\Gamma\left(\frac{11}{8}\right)\Gamma\left(\frac{1}{2}\right)} \frac{\mu^{14}}{R_E^{14}} \frac{1}{L_i^7\, L^7\, (\tau')^{15/8}} \exp - \left[\frac{\mu^8}{64\, R_E^8\, \tau'} \left(\frac{1}{L_i^8} + \frac{1}{L^8}\right)\right]$$

$$\int_{-1}^{+1} dx\, \left(1 - x^2\right)^{3/8} \exp\left(\frac{\mu^8}{32\, R_E^8\, L_i^4} \frac{x}{L^4\, \tau'}\right) . \qquad (31)$$

In Figure 2 we display results of numerically evaluating Equation 31 with $L_i = 8$ as previously. $\langle \overline{Q} \rangle$ is once more in arbitrary units with the same normalization as in Figure 1, so that direct comparison of the figure is possible. Note, however, that in this case $T' = 190\, L_i^8\, R_E^8\, \tau' / \mu^8 = .043\, t$ (in hours), so that $T' = 1$ corresponds to approximately 23 hours of real time t. For values $\tau_c = 1$ hr., $\mathcal{A} = \left(2 \times 10^{-4}\ \mathrm{V/m}\right)^2$, T' is thus roughly 4/3 times greater than T which parameterizes Figure 1.

Qualitative features of Figure 2 are similar to those of Figure 1: the peak in $\langle \overline{Q} \rangle$ moves inward with time; $\langle \overline{Q} \rangle$ drops off more sharply on its low L side than in its high L side; and there is a net flux of particles to $L = \infty$. The position of L_{max} as determined from an asymptotic analysis of Equation 31 similar to that carried out in Appendix B for Equation 24 is

$$L_{max} = 8\left\{1 - 1.6 \times 10^{-2}\, T' + O\left[(T')^2\right]\right\} \qquad T' \ll 1 , \qquad (32a)$$

$$L_{max} = \frac{8}{(.3T')^{1/8}} \left\{1 - \frac{.2}{T'} + O\left[(T')^{-2}\right]\right\} \qquad T' \gg 1 . \qquad (32b)$$

The velocity of the $\langle \bar{Q} \rangle$ peak in the two asymptotic regimes is

$$\frac{dL_{max}}{dt} = -.13 \frac{dT'}{dt} \left[1 + O(T')\right] \qquad T' \ll 1$$

$$= -5.6 \times 10^{-3} \left[1 + O(t)\right] \frac{R_E}{hr} \qquad t \ll 23 \text{ hrs}, \qquad (33a)$$

$$\frac{dL_{max}}{dt} = -\frac{1}{(.3T')^{1/8}} \frac{1}{T'} \frac{dT'}{dt} \left\{1 + O\left[(T')^{-1}\right]\right\} \qquad T' \gg 1$$

$$= -\frac{1.7}{t^{9/8}} \left[1 + O\left(t^{-1}\right)\right] \frac{R_E}{hr} \qquad t \gg 23 \text{ hrs}. \qquad (33b)$$

As a final step we compare the velocities dL_{max}/dt for the two mechanisms, calculating the ratio R of the rates predicted by Equations 27 (where the diffusing mechanism is the convection electric field) to the Nakada-Mead rate Equations 33. In the long and short time asymptotic limits R is respectively

$$R = \frac{4.8 \times 10^{6} \tau_c \mathcal{a}}{5.6 \times 10^{-3}} \approx 34 \qquad t \ll 18 \text{ hrs} \qquad (34a)$$

$$R = \frac{.039}{(\tau_c \mathcal{a})^{1/4} t^{5/4}} \frac{t^{9/8}}{1.7} \approx \frac{1.6}{t^{1/8}} \qquad t \gg 18 \text{ hrs} \qquad (34b)$$

In computing R, values $\tau_c = 1$ hr., $\mathcal{a} = \left(2 \times 10^{-4} \text{ V/m}\right)^2$ have been used.

Equations 34 express analytically a significant feature of Figures 1 and 2: at times when $\langle \bar{Q} \rangle$ is sufficiently peaked (at an L value where diffusion is not inhibited by a strong magnetic field), L_{max} in the case of convection electric fields moves a good order of magnitude faster than L_{max} in the situation where surface

currents are the agent. While Equation 34b does predict that the role of the current variation mechanism eventually exceeds that of convection electric fields, it is evident from the figures that this occurs only after L_{max} has moved inward considerably and its velocity of motion is measurably reduced from the early fast moving stage of its evolution. Compare also the height of the maxima in Figures 1 and 2 for the same value of L: as time advances the maximum value of $\langle \overline{Q} \rangle$ in Figure 2 is down from that in Figure 1 by over an order of magnitude.

The features mentioned in the previous paragraph are the basis for our conclusion that convection electric fields can move radiation trapped in the magnetosphere radially inward at a rate at least an order of magnitude more rapidly than electric fields of magnetopause origin.

CONCLUDING REMARKS

While our results are encouraging in their prediction that convection electric fields can possibly be a strong diffusing force on trapped magnetospheric radiation, it would be hazardous to conclude that the mechanism presented here is the only agent. Such a conclusion is especially unwarranted in view of the small amount of evidence for either plasma flow or large scale electric fields in the magnetosphere. It may well be that a combination of effects are operative: diffusion by magnetic variations complementing the convection mechanism near the magnetopause and diffusion due to drift instabilities and even non-adiabatic processes playing a prominent part at plasmapause distances and closer in.

In view of the scarcity of electric field measurements, an enlightening experimental exercise is to analyze statistically plasma flow and magnetic field in the magnetosheath. The autocorrelation time and mean square deviation for both these quantities is information from which corresponding properties of the magnetospheric electric field might be inferred.

On the theoretical side, the addition of plausible loss mechanisms and solution of either the ensuing equilibrium or time dependent transport equations are needed before comparison with experimental flux profiles becomes significant. Depending upon the L and $\langle \tilde{Q} \rangle$ variation of the diffusion coefficients, different solutions to the transport equation are expected. Comparison of such results with experiment may then further enhance the credibility of particular mechanisms both for inwardly diffusing particles and scattering them into the loss cone.

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APPENDIX A

DERIVATION OF THE α DIFFUSION EQUATION

A derivation of the one dimensional diffusion Equation 5 is presented here. Our starting point is the two dimensional diffusion equation

$$\frac{\partial \langle Q \rangle (\alpha, \beta, M, J, t)}{\partial t} + \langle \dot{\alpha} \rangle \frac{\partial \langle Q \rangle}{\partial \alpha} + \langle \dot{\beta} \rangle \frac{\partial \langle Q \rangle}{\partial \beta}$$

$$= \frac{\partial}{\partial \alpha} \left(D_{\alpha\alpha} \frac{\partial \langle Q \rangle}{\partial \alpha} \right) + \frac{\partial}{\partial \alpha} \left(D_{\alpha\beta} \frac{\partial \langle Q \rangle}{\partial \beta} \right) + \frac{\partial}{\partial \beta} \left(D_{\beta\alpha} \frac{\partial \langle Q \rangle}{\partial \alpha} \right)$$

$$+ \frac{\partial}{\partial \beta} \left(D_{\beta\beta} \frac{\partial \langle Q \rangle}{\partial \beta} \right) \qquad \text{(A-1)}$$

derived by Birmingham, Northrop, and Fälthammar (1967). Equation A-1 describes the evolution of the density $\langle Q \rangle$ in an α, β, M, J phase space, of particles acted upon by an ensemble of electric and magnetic forces each realization of which varies in time in an erratic fashion. The density $\langle Q \rangle$ has been averaged over this ensemble of fields. Effects of the driving forces are included in the ensemble averaged velocities $\langle \dot{\alpha} \rangle$ and $\langle \dot{\beta} \rangle$ as well as in the diffusion coefficients $D_{\alpha\alpha}$, $D_{\alpha\beta}$, $D_{\beta\alpha}$, and $D_{\beta\beta}$. Since M and J are conserved during interaction with the fields, diffusion only occurs in the subspace of the Euler potentials α and β. For the dipole magnetic field we choose $\alpha = - \mu \sin^2 \vartheta / r$ and $\beta = \phi$.

To obtain the one dimensional α-equation, we first average (A-1) over $\beta(=\phi)$ at fixed α, M, and J. Denoting this β-average by a bar ($^-$), we thus obtain

$$\frac{\partial \langle \bar{Q} \rangle (\alpha, M, J, t)}{\partial t} + \frac{\partial}{\partial \alpha}\left(\overline{\langle \dot{\alpha} \rangle \langle Q \rangle}\right) = \frac{\partial}{\partial \alpha}\left(\overline{D_{\alpha\alpha} \frac{\partial \langle Q \rangle}{\partial \alpha}}\right)$$

$$+ \frac{\partial}{\partial \alpha}\left(\overline{D_{\alpha\beta} \frac{\partial \langle Q \rangle}{\partial \beta}}\right) \qquad \text{(A-2)}$$

Here use has been made of 1) the periodicity in β of all physical quantities and 2) the canonical nature,

$$\frac{\partial \langle \dot{\alpha} \rangle}{\partial \alpha} + \frac{\partial \langle \dot{\beta} \rangle}{\partial \beta} = 0 , \qquad \text{(A-3)}$$

of α and β as guiding center coördinates (Northrop, 1963).

Equation A-1 is correct through $O(\epsilon\delta^2)$ in the $O(\epsilon = m/e)$ adiabatic expansion parameter and the $O(\epsilon\delta)$ $(\epsilon \ll \delta \ll 1)$ smallness of the driving fields. We next assume that $\langle Q \rangle$ deviates from β-homogeneity only by an amount proportional to δ, i.e., $\Delta Q = \langle Q \rangle - \langle \bar{Q} \rangle = O(\delta)$. This situation is satisfied, for example, if $\langle Q \rangle (t = 0)$ is β-independent, since inhomogeneities in β then appear only as a consequence of the $O(\epsilon\delta)$ fields acting for a time $(\sim\epsilon^{-1})$ of the order of a particle drift. As a result of this assumed near homogeneity in β, Equation A-2 simplifies to

$$\frac{\partial \langle \bar{Q} \rangle}{\partial t} + \frac{\partial}{\partial \alpha}\left(\overline{\langle \dot{\alpha} \rangle \langle Q \rangle}\right) = \frac{\partial}{\partial \alpha}\left(\overline{D}_{\alpha\alpha} \frac{\partial \langle \bar{Q} \rangle}{\partial \alpha}\right) . \qquad \text{(A-4)}$$

To obtain the desired result, Equation 5, we must now show that

$$\frac{\partial}{\partial \alpha}\left(\overline{\langle \dot{\alpha} \rangle \langle Q \rangle}\right) = \frac{\partial}{\partial \alpha}\left[\langle \bar{\dot{\alpha}} \rangle \langle \bar{Q} \rangle + \overline{\Delta \langle \dot{\alpha} \rangle \Delta \langle Q \rangle}\right] \tag{A-5}$$

is of a negligible order of smallness. The canonical nature (Northrop, 1963) of the α, β coordinates is next used to show

$$\langle \bar{\dot{\alpha}} \rangle = -\frac{c}{e}\, \frac{\partial \overline{K(\alpha, \beta, J, M, t)}}{\partial \beta} = 0 . \tag{A-6}$$

For our model in which no magnetic variations occur, the Hamiltonian $\bar{K}$ is the total particle energy, kinetic plus potential.

We are thus left with indicating the smallness of $\partial/\partial \alpha \left[\overline{\Delta \langle \dot{\alpha} \rangle \Delta \langle Q \rangle}\right]$. If in Equation 2 for the electric potential, $\langle A \rangle = 0$, we could straightforwardly argue that $\Delta \langle \dot{\alpha} \rangle$, then at most $O\left(\epsilon^2 \delta^2\right)$ (no $\langle \dot{\alpha} \rangle$ arises from the dipole field alone) coupled with $\Delta Q \sim O(\delta)$ leads to a negligibly small $O\left(\epsilon^2 \delta^3\right)$ term. The assumption that $\langle A \rangle = 0$, however, means that the probability of a dawn-dusk directed equatorial electric field is the same as a dusk-dawn directed one. We feel that such equal probability is in contradiction with the extant circumstantial evidence and hence we here attempt to prove the smallness of $\partial/\partial \alpha \left[\overline{\Delta \langle \dot{\alpha} \rangle \Delta \langle Q \rangle}\right]$ without the benefit of $\langle A \rangle = 0$.

Using Equations A-1 and A-4 it is readily shown that

$$\frac{\partial\left(\Delta \langle Q \rangle\right)}{\partial t} + \omega_D \frac{\partial\left(\Delta \langle Q \rangle\right)}{\partial \beta} = -\Delta \langle \dot{\alpha} \rangle \frac{\partial \langle \bar{Q} \rangle}{\partial \alpha} + O\left(\epsilon \delta^2\right) , \tag{A-7}$$

where ω_D is the longitudinal drift rate in the absence of E. Equation A-7 has the solution

$$\Delta\langle Q\rangle = \psi\left(\alpha, \beta - \omega_D t, M, J\right)$$

$$- \int_0^t dt' \, \Delta\langle\dot{\alpha}\rangle \left[\alpha, \beta + \omega_D (t' - t), M, J, t'\right] \frac{\partial\langle\bar{Q}\rangle}{\partial\alpha} (\alpha, M, J, t') \, , \qquad \text{(A-8)}$$

$\psi(\alpha, \beta, M, J)$, the initial value of $\Delta\langle Q\rangle$, being henceforth taken equal to zero. Employing the relationship (BNF, 1967)

$$\Delta\langle\dot{\alpha}\rangle = - c \frac{\partial\Delta\langle V\rangle}{\partial\beta} \qquad \text{(A-9)}$$

and the explicit form Equation 2 for the potential, we find

$$\frac{\partial}{\partial\alpha}\left[\overline{\Delta\langle\dot{\alpha}\rangle \, \Delta Q}\right] =$$

$$- \frac{\langle A\rangle^2 \mu^2 c^2}{2} \frac{\partial}{\partial\alpha}\left\{\frac{1}{\alpha^2} \int_0^t dt' \cos\omega_D (t' - t) \frac{\partial\langle\bar{Q}\rangle}{\partial\alpha} (\alpha, M, J, t')\right\} . \qquad \text{(A-10)}$$

$\langle A\rangle$ has been assumed to be independent of time.

The presence of the factor $\cos\omega_D (t' - t)$ oscillating over the drift period together with the slowly varying $\partial\langle\bar{Q}\rangle/\partial\alpha$ render $\partial/\partial\alpha\left[\overline{\Delta\langle\dot{\alpha}\rangle \Delta Q}\right]$ negligible as a driving term in Equation A-4. This may be seen formally by averaging

Equation A-10 over a drift period τ_D:

$$\frac{1}{\tau_D}\int_T^{T+\tau_D} dt\, \frac{\partial}{\partial \alpha}\left[\overline{\Delta\langle\dot{\alpha}\rangle\,\Delta Q}\right]$$

$$= -\frac{\langle A\rangle^2 \mu^2}{2}\,\frac{c^2\,\tau_D^2}{(2\pi)^2}\,\frac{\partial}{\partial\alpha}\left\{\frac{1}{\alpha^2}\,\frac{\partial^2\langle\bar{Q}\rangle}{\partial\alpha\,\partial T}\left[1 + O\left(\frac{\tau_D}{\tau_d}\right)\right]\right\} \qquad \text{(A-11)}$$

where $O(\tau_D/\tau_d)$ correction terms in (A-11) are small in the ratio of the drift time τ_D (an ϵ^{-1} time) to the diffusion time τ_d (an $\epsilon^{-1}\delta^{-2}$ time). The leading order term in (A-11) is itself of $O(\tau_D/\tau_d)$ in smallness compared with the diffusion term on the right hand side of Equation 5. Our conclusion is then that all terms represented by Equation A-11 are of a negligible $O(\epsilon\delta^4)$ order of smallness and that (A-4) may be written

$$\frac{\partial\langle\bar{Q}\rangle}{\partial t} = \frac{\partial}{\partial\alpha}\left(\bar{D}_{\alpha\alpha}\,\frac{\partial\langle\bar{Q}\rangle}{\partial\alpha}\right). \qquad \text{(A-12)}$$

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APPENDIX B

THE ASYMPTOTIC MOTION OF L_{max}

We here calculate asymptotically as a function of time the L value at which $\langle \bar{Q} \rangle$, as given by Equation 24, maximizes. An analogous procedure has also been applied to Equation 31 but this calculation is not detailed here.

For present considerations normalization of $\langle \bar{Q} \rangle$ is unimportant and Equation 24 is written

$$\langle \bar{Q} \rangle = \frac{1}{L^3 \tau^{7/4}} \exp - \left[\frac{\mu^4}{16 R_E^4 \tau} \left(\frac{1}{L_i^4} + \frac{1}{L^4} \right) \right]$$

$$\int_{-1}^{+1} dx \left(1 - x^2\right)^{1/4} \exp \left(\frac{\mu^4}{8 R_E^4 L_i^2 L^2} \frac{x}{\tau} \right) \qquad \text{(B-1)}$$

L_{max} is then determined from the equation

$$L \frac{\partial}{\partial L} \left(\ell n \langle \bar{Q} \rangle \right) = \left(-3 + \frac{\mu^4}{4 R_E^4 \tau} \frac{1}{L^4} \right)$$

$$- \frac{\mu^4}{4 R_E^4 L_i^2 L^2 \tau} \frac{\displaystyle\int_{-1}^{+1} dx\, x \left(1 - x^2\right)^{1/4} \exp \frac{\mu^4 x}{8 R_E^4 L_i^2 L^2 \tau}}{\displaystyle\int_{-1}^{+1} dx \left(1 - x^2\right)^{1/4} \exp \frac{\mu^4 x}{8 R_E^4 L_i^2 L^2 \tau}} = 0 \,. \qquad \text{(B-2)}$$

We see no way to obtain analytically an exact solution to Equation B-2. However, progress is possible in the limits $\beta = \mu^4/8 R_E^4 L_i^2 L^2 \tau \gg 1, \ll 1$ when the

third term in (B-2),

$$I = -2\beta \frac{\partial}{\partial\beta} \left\{ \ell n \left[\int_{-1}^{+1} dx \, (1-x^2)^{1/4} \exp \beta x \right] \right\}$$

$$= -2\beta \frac{\partial}{\partial\beta} (\ell n\, G) \qquad \text{(B-3)}$$

can be asymptotically evaluated.

Consider first the short time $\tau \to 0$, $\beta \to \infty$ limit. In this case we introduce the change of variable $x = 1 - y$ and write

$$G = \exp(\beta) \int_0^2 dy \, y^{1/4} (2-y)^{1/4} \exp - \beta y \, . \qquad \text{(B-4)}$$

As $\beta \to \infty$, the only contribution to the y-integral comes from the region about $y = 0$, where we expand

$$(2-y)^{1/4} = (2)^{1/4} \left[1 - \frac{y}{8} + O(y^2) \right] . \qquad \text{(B-5)}$$

Substituting this expansion into Equation B-4, we perform the integrations and obtain

$$G = \frac{(2)^{1/4} \exp \beta}{\beta^{5/4}} \left[\gamma\left(\frac{5}{4}, 2\beta\right) - \frac{1}{8\beta} \gamma\left(\frac{9}{4}, 2\beta\right) + O(\beta^{-2}) \right] . \qquad \text{(B-6)}$$

Here $\gamma(a, x)$ is the incomplete gamma function (Gradshteyn and Ryzhik, 1965). For large x, γ has the asymptotic form

$$\gamma(a, x) = \Gamma(a) + O\left(x^{a-1} \exp - x\right) \qquad (x \to \infty) \tag{B-7}$$

where Γ is the ordinary gamma function. In the limit $\beta \to \infty$, G thus becomes

$$G = \frac{(2)^{1/4} \exp \beta}{\beta^{5/4}} \left[\Gamma\left(\frac{5}{4}\right) - \frac{1}{8\beta}\Gamma\left(\frac{9}{4}\right) + O\left(\beta^{-2}\right)\right], \tag{B-8}$$

and

$$I = -2\beta\left[1 - \frac{5}{4\beta} + \frac{5}{32\beta^2} + O\left(\beta^{-3}\right)\right]. \tag{B-9}$$

Substituting (B-9) into (B-2) and identifying β, we finally write down the equation

$$-3 + \frac{\mu}{4 R_E^4 \tau}\left(\frac{1}{L^4} - \frac{1}{L_i^2 L^2}\right) + \frac{5}{2} + O\left(\frac{8 R_e^4 L_i^2 L^2 \tau}{\mu^4}\right) = 0 . \tag{B-10}$$

The asymptotic solution to Equation B-10 is

$$L_{max} = L_i\left\{1 - \frac{R_E^4 L_i^4}{\mu^4}\tau + O\left[\left(\frac{R_E^4 L_i^4}{\mu^4}\tau\right)^2\right]\right\} \tag{B-11}$$

in the limit $R_E^4 L_i^4 \tau/\mu^4 \ll 1$.

In the opposite, long time, asymptotic limit, $\tau \to \infty$, $\beta \to 0$, we approximate G by

$$G = \int_{-1}^{+1} dx \left(1 - x^2\right)^{1/4}\left[1 - \beta x + \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{6} + O\left(\beta^4\right)\right]. \tag{B-12}$$

Contributions arise only from even powers of x in the series. Performing the integrations, we find

$$G = \frac{(2)^{1/2}}{6} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \left\{1 + \frac{\beta^2}{7} + O\left(\beta^4\right)\right\} , \tag{B-13}$$

so that in this limit

$$I = -\frac{4\beta^2}{7} + O\left(\beta^4\right) . \tag{B-14}$$

L_{max} is hence determined from the equation

$$-3 + \frac{\mu^4}{4 R_e^4 \tau} \frac{1}{L^4} - \frac{1}{112} \frac{\mu^8}{R_E^8 \tau^2 L_i^4 L^4} + O\left[\left(\frac{\mu^4}{R_E^4 \tau L_i^2 L^2}\right)^3\right] = 0 , \tag{B-15}$$

whose solution yields

$$L_{max} = \frac{\mu}{R_E (12\tau)^{1/4}} \left\{1 - \frac{\mu^4}{112 R_E^4 L_i^4 \tau} + O\left[\left(\frac{\mu^4}{R_E^4 L_i^4 \tau}\right)^2\right]\right\} . \tag{B-16}$$

FIGURE CAPTIONS

Figure 1: $\langle \overline{Q} \rangle$, as given by Equation 24, for a δ-function at $L_i = 8$ at $T = 0$. Diffusion is driven by convection electric fields, and for representative magnetospheric conditions $T = 1$ unit corresponds to 18 hrs. of real time.

Figure 2: $\langle \overline{Q} \rangle$ from Equation 31 for an $L_i = 8$, δ-function initial condition. The diffusion coefficient is in this case taken from the work of Nakada and Mead (1965). $T' = 1$ unit corresponds to 23 hrs. of real time.

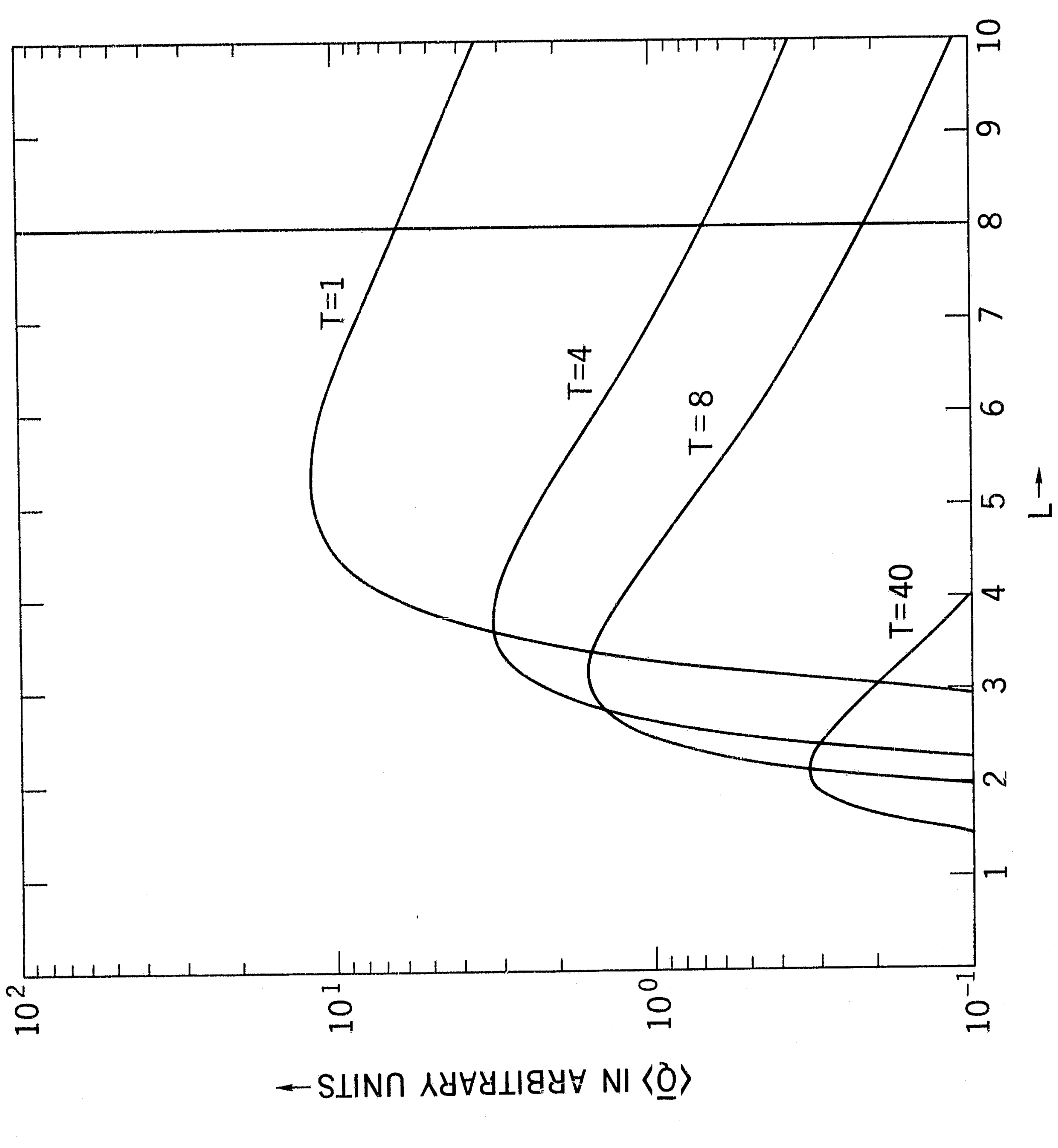

Figure 1

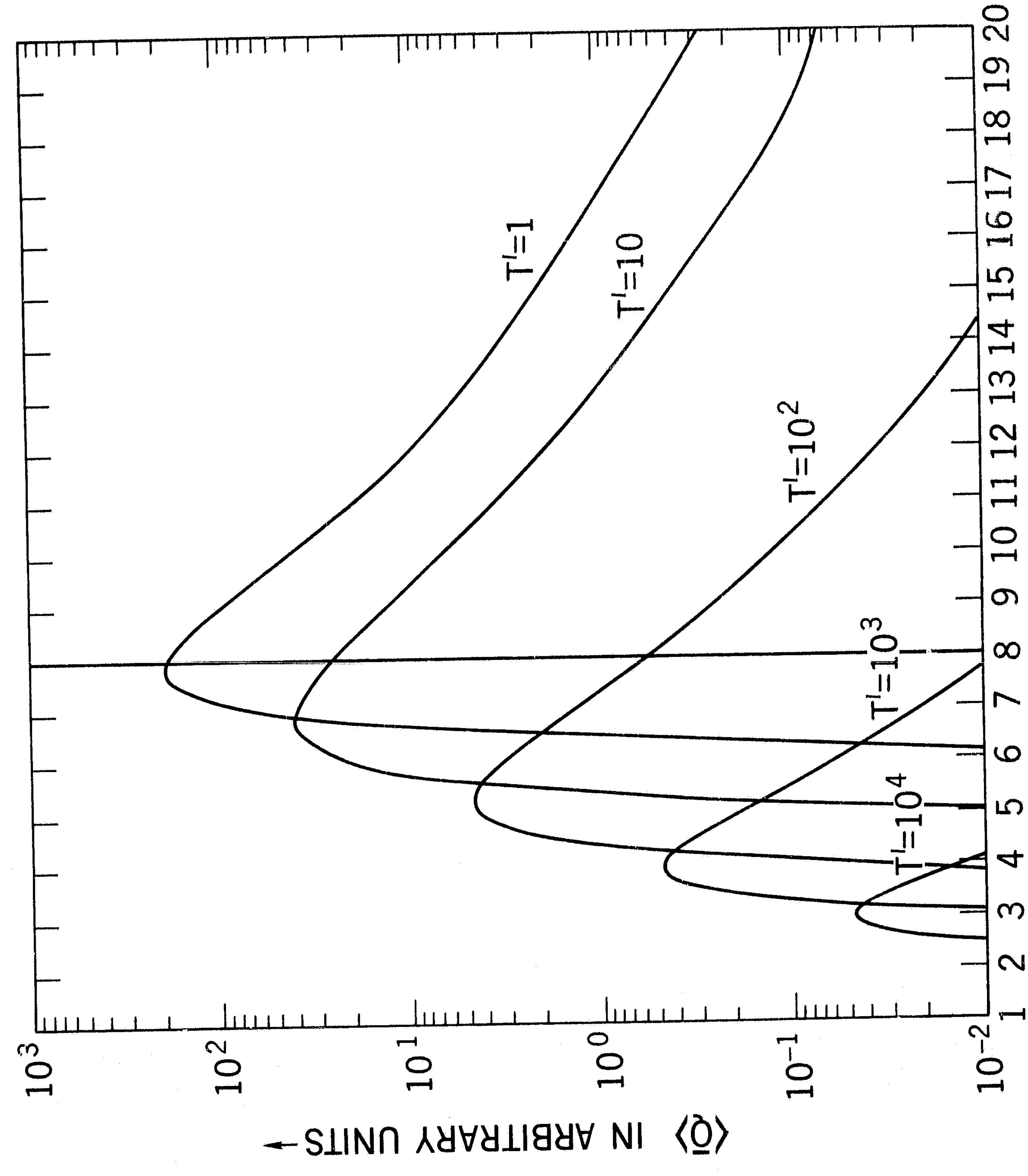
⟨Q⟩ IN ARBITRARY UNITS →
L →
T'=1
T'=10
T'=10^2
T'=10^3
T'=10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Figure 2